# Multilateration Techniques for a Short Range Lightning Location System

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Abstract — Storm Detector device [1] demonstrated that a low-cost lightning detector – The integrated circuit AS3935 made by Austria Microsystems – has a detection efficiency (DE) of about 76% and a Probability of Detection (POD) of about 80%, both for a radius of 20 km and also deduced an expression for the event distance given the detected event parameters, with a maximum error of 3 km relative to the truth data given by LINET network [2]. This article presents a simulation of multilateration techniques of a hypothetical network with 19 Storm Detector devices, using generated lightning data, giving some insights on how it will work, using DE and Location Accuracy (LA) as parameters. DE parameter reached 100% in 45.5% of the lightning area, the maximum location error in the simulation was improved by  $\approx 36\%$  relative to one sensor/device and the Region of Lightning Occurrence (RLO), defined as the area of maximum probability of lightning position.

# I. INTRODUCTION

The Storm Detector device (SDD) [1] developed by the Atmospheric Electricity Phenomena Group (FEA/UFPR) is an apparatus initially targeted for research purposes, equipped with an AS3935 sensor [3] for lightning detection, controlled by an ESP32 [4], a System-On-Chip (SoC) microprocessor. It also embeds a GPS device for location/time synchronization and a small storage device for data logging. The ESP32 also has several embedded features, like WiFi and Bluetooth, which makes it suitable for standard Internet of Things (IoT) applications [4]. The SDD firmware was improved to accommodate network services, like a remote storm and geolocation data gathering through Message Queueing Telemetry Transport (MQTT) protocol [5] and sensor control, diagnostics, configuration and time synchronization, in case of GPS malfunction. Some additions, like operation over LoRaWAN networks [6], especially important when operating in remote areas, are planned to take place until the middle of 2021. FEA/UFPR has built 11 SDD devices that will be part of an experimental short-range lightning location network, still in deployment to this date.

Lightning Location Systems (LLS) data is used to access storm risks and compute lightning flash density (Ng) [7]. It is also used in real-time by meteorological communities to forecast severe weather [8], [9]. The LLS performance parameters are defined to at least Detection Efficiency (DE), Location Accuracy(LA) and Classification Accuracy(CA) [7] Although it is now a research device, Storm Detector was also designed to be a low-cost alternative to access storm risks and alerts in wide areas where communication and/or financial resources are a great issue, like rural areas, remote electrical power installations, oil refineries, airports, tourist areas and populations close to mountainous regions, offering an acceptable performance to such tasks, like a reasonable LA and real-time operation [1].

Mialdea-Flor et al [10] demonstrated, with a network-based on AS3539 sensors, that lateration techniques are possible with these devices. The 2019/2020 summer season was poor in lightning events, but it was possible to corroborate [10] on one occasion, with 5 SDD devices installed around Curitiba city - Brazil. The test network captured 25 events, 20 of which was good for multilateration.

The current mainline of the FEA/UFPR group regarding SDD devices is to build a reliable, responsive, high DE and LA real-time short-range lightning location network, aimed at serving poorly attended regions regarding severe weather and lightning strikes. Build a network is a huge task, and several algorithms were tested with different simulation scenarios to certify its operation. One of them is presented in this article: the choice of a multilateration technique to be the best candidate for lightning location algorithm.

# **II. THEORETICAL FOUNDATIONS**

The determination of an unknown moving or static point in space from known locations is called multilateration [11]–[14]. The multilateration problem solves a set of distance equations from an unknown point A to known points, called stations,  $S_i$ . There are various algorithms and closed forms for solving multilateration problems, as the atomic multilateration [15], Bancroft's Algorithm [16], Razin's Algorithm [16] and some authors found closed forms to three and four slant ranges [11], [16], [17]. However, there are iterative methods for multilateration solution [16].

The euclidean distance from any point A to a known point  $S_i$ , is given as:

$$\delta(x_i) = \|x_i\| \tag{1}$$

Where  $x_i$  is the vector from A to  $S_i$ . Given  $\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n))^T$  as the distance vector and  $\hat{\delta}(x) = (\hat{\delta}_1, \hat{\delta}_2, \dots, \hat{\delta}_n)^T$  as the measurements vector, a residual function f(x) is defined by:

$$f(x) = \delta(x) - \hat{\delta}(x) \tag{2}$$

The solution of f(x) = 0 is the solution of a multilateration problem. When using perfect measurements, the solution is trivial, but in many cases are assumed that the

distance measurements are corrupted by noise or imprecisions. So, a trivial solution of f(x) is almost impossible, but it can be estimated as a region in space where probably the solution would be. Then, the solution of f(x) can be described as an optimization problem.

#### A. Iterative Methods for Multilateration

An optimization problem is defined by finding a region or a set of values that maximize or minimize a model function [14], [18] and in many cases involves a solution of leastsquares problem. Considering the model function (2),

$$F(x) = \frac{1}{2} \left\| f(x) \right\|^2$$
(3)

as the objective function and assuming that the descending condition at iteration k is satisfied

$$F(x_{k-1}) > F(x_k) \tag{4}$$

the optimization problem is to find the solution  $x^*$  that satisfies a minimization rule:

$$x^* = \operatorname{argmin}_x(F(x)) \tag{5}$$

In the methods described in this article, a general framework to solve the minimization problem can be oulined as [18]:

# $x := x_0$ k = 0repeat

Find a descent direction  $\lambda_k$ 

$$x := x + \lambda_k$$
$$k := k + 1$$

#### until Stop

Where k is used as one of the stop conditions (maximum iterations) and  $x_0$  is an initial guess of the solution. At least 3 methods for solving the step  $\lambda_k$  are found in literature [16], [18]–[23]:

a) Steepest Descent Gradient Algorithm (SDG): consists in to make the step  $\lambda_k$  proportional of the negative gradient of F(x):

$$\lambda_k = -\mu \nabla F(x_k) \tag{6}$$

Where  $\mu$  is any value from 0 to 1 and the gradient is written as

$$\nabla F(x_k) = J(x_k)^T f(x_k). \tag{7}$$

When  $\mu = 1$  it operates at the steepest gradient. This method is of fast convergence when  $x_k$  is far from the solution, but it has poor convergence properties when  $x_k$  comes close to the solution.

b) Gauss-Newton Algorithm (GNA): the Newton's step to find the extremes of F(x) is the solution of

$$\nabla^2 F(x_k)\lambda_k = -\nabla F(x_k) \tag{8}$$

and serves as a basis to the Gauss-Newton method [18], [21].  $\nabla^2 F(x_k)$  is the Hessian matrix of  $F(x_k)$ , written as

$$\nabla^2 F(x_k) = J^T(x_k) J(x_k) + \sum_{i=1}^m f_i(x_k) \nabla^2 f_i(x_k)$$
(9)

where  $J(x_k)$  is the Jacobian matrix of  $f(x_k)$  and *m* is the rank of  $J(x_k)$ .

Newton's method is of fast convergence when  $x_k$  is close to the solution but is not robust, especially when  $J(x_k)$  is not full rank. Gauss-Newton is based on the Taylor expansion of f(x) [18], [21]:

$$f(x_{k+1}) \simeq l(\lambda_k) \equiv f(x_k) + J^T(x_k)\lambda_k$$
(10)

An approximation of F(x) is also possible, as:

$$F(x_{k+1}) \simeq L(\lambda_k) \equiv \frac{1}{2} l^T(\lambda_k) l(\lambda_k)$$
(11)

The second term of  $\nabla^2 F(x_k)$  is negligible when  $x_k$  is close enough to the solution and the Hessian matrix can be approximated by:

$$\nabla^2 F(x_k) \simeq H(x_k) = J^T(x_k)J(x_k) \tag{12}$$

and then the step  $\lambda_k$  is the minimizer of  $L(\lambda_k)$ , given as the solution of

$$H(x_k)\lambda_k = -\nabla F(x_k). \tag{13}$$

Gauss-Newton is of fast convergence when  $x_k$  is close to the solution, but it has poor convergence when  $x_k$  is far from the optimal values. It also has the lack of robustness of Newton's method [18].

c) Levenberg-Marquardt Algorithm (LMA): this method is an interpolation between Gauss-Newton and Steepest Descent Gradient methods. Proposed by [19] and [20], the method introduces one term in Gauss-Newton update [18], given as:

$$(H(x_k) + \mu_k I)\lambda_k = -\nabla F(x_k) \tag{14}$$

Where  $\mu_k$  is the damping parameter of the algorithm step, a strictly positive value. When  $\mu_k$  is small, the step behaves like a Gauss-Newton update and when  $\mu_k$  is large it behaves like a gradient descent update. The value of  $\mu_k$  at first iteration is usually large, and  $\mu_k$  is expected to decrease in the subsequent iterations. The parameter  $\mu_k$  is evaluated each iteration, and [18] proposed an update rule for it. The gain factor  $\vartheta_k$  is given as:

$$\vartheta_k = \frac{F(x_{k-1}) - F(x_k)}{L(0) - L(\lambda_k)}$$
(15)

and represents how good the approximation  $L(\lambda_k)$  is to  $F(x_k + \lambda_k)$ . Then update  $\mu_k$  according to  $\vartheta_k$ , as oulined in the pseudocode below [18].

if  $\vartheta_k > 0$  then

$$\mu_{k} := \mu_{k} \cdot \operatorname{argmax} \left( \frac{1}{\gamma}, 1 - (\beta - 1)(2\vartheta_{k} - 1)^{\rho} \right)$$

$$v := \beta$$
else
$$\mu_{k} := \mu_{k} v$$

$$v := 2v$$
end if
Where  $v_{0} = \beta = 2$  and  $\gamma = \rho = 3$ . The initial value of  $\mu$  is given by [18]:
$$\mu_{0} = \tau \cdot \operatorname{argmax}(\operatorname{diag}(J^{T}(x_{0})J(x_{0})))$$
(16)

with  $\tau = 0.8$ , which is adequate to our multilateration problem.

For all algorithms, it was used

$$\|F(x_k) - F(x_{k-1})\| < \varepsilon \tag{17}$$

as the convergence criterion.

#### B. Initial Solution $x_0$

The iterative methods rely on a good initial guess of the solution. A simple trilateration algorithm (TRL) as the proposed by [10], which is well described in [16] and [17] is used to provide the initial guess.

#### C. Geometric Dilution Of Precision (GDOP)

Geometric Dilution of Precision (GDOP) is the effect of the spatial relationship of the stations  $S_i$  relative to the point *A*. The quality of the solution is affected by that geometry and is referred to as Dilution of Precision (DOP) [24], [25]. The higher the DOP is, the more inaccurate is the obtained solution. It is possible, given a limit for DOP values, to design an array of sensors/stations to give more accurate solutions [25].

The measurements are considered to have the same characteristics, so, the measurement errors are considered uncorrelated and have the same variance  $\sigma_{meas}^2$  [16], [24], [25]. The covariance of the estimated point  $x^*$  is given as [16]:

$$E[(x^* - x)(x^* - x)^T] = H(x^*)^{-1}\sigma_{meas}^2$$
(18)

The matrix  $H(x^*)^{-1}$  contains all geometric information about the solution [16].  $M(x^*) = C(x^*)^T H(x^*)^{-1} C(x^*)$  is a dimensionless and rotated version of  $H(x^*)^{-1}$  matrix, where  $C(x^*)$  denotes a rotation matrix to the local-level coordinates frame relative to the point *A* estimation [16]. The GDOP value  $\xi(x^*)$  is the square root of the trace of  $M(x^*)$ , given as

$$\xi(x^*) = \sqrt{\sum_{i=1}^{n} M(x^*)_{ii}}$$
(19)

where  $M(x^*)_{ii}$  represents the principal diagonal elements and is used as the quality factor of the solution. The standard deviation of the estimated position  $\sigma_{x^*}$  is given as

$$\sigma_{x^*} = \sigma_{meas} \xi(x^*) \tag{20}$$

where  $\sigma_{meas}$  is the standard deviation of the distance measures. In this perspective, GDOP can be interpreted as a scale factor between position and measurements standard deviations.

#### D. Region of Lightning Occurrence (RLO)

Region of Lightning Occurrence (RLO) is the area of maximum probability of lightning occurrence given a location estimation. The radius of RLO is defined by the confidence level given in the percentage of the location error distribution. The importance of RLO consists in delimiting a search area for lightning hazards, given that lightning has occurred. It is of particular interest to electric power companies. A formal definition of RLO radius ( $\phi(c)$ ) is given as:

$$\phi(c) = Q(c) \tag{21}$$

i.e, according to the definition, RLO radius is the quantile value of location error distribution at a confidence level c. The RLO itself is at a first approximation the area of a circle of the same radius:

$$W(c) = \pi \phi^2(c) \tag{22}$$

# **III. SIMULATION SETUP**

# A. Sensor simulation

Heilmann et al [1] has made an extensive study about AS3935 Lightning sensor parameters. It found an expression for detection efficiency as a function of distance  $\delta$  for Storm Detector device, given as:

$$ADE_{SD}(\delta) = 88.11 \left( 1 - \frac{1}{1 + 309.6e^{-0.2888\delta}} \right)$$
(23)

It also determined the location error of Storm Detector device, given as:

$$e(x) = \mathcal{N}(-0.26, 1.84^2) \tag{24}$$

where e(x) is given in km.

To use (23) the simulation process uses a simple Bernoulli trial, using the output of a random generator to compare with the detection probability given by (23) for the point distance from a given sensor. This procedure also enables simulation of the Network Detection Efficiency (NDE) for a given point in space. A detection efficiency simulation was made and is presented in the results section. In this simulation, a successful detection is when 3 or more sensors detected the same event.

The location error given by (24) was added to every sensor distance as gaussian noise. There are, due to noise, measurements that can be statistically eliminated because they don't help to find the solution and are called outliers. The simulation purges the measurements vector every 10 iterations to eliminate outliers, doing a z-test [26] and eliminates the measurements with z > 3.0.

# B. General Setup

The iterative algorithms SGD, GNA and LMA, among other things needed for the simulation, were implemented using Python language, version 3.8 [27]. It also uses Numba Jit Compiler [28] to improve performance. The hypothetical network tested in this simulation is pictured in Fig. 1.



Fig. 1. Sensor disposition for simulation

The black dots are the sensor positions. The sensor disposition was chosen considering a detection range of 25 km and the geometry is to minimize geometrical effects on the solution [25]. The network has a detection radius of around 37.2 km with a DE of at least 95%.

The lightning dataset was generated by filling homogeneously a square of 9554.56 km<sup>2</sup> around the network with lightning locations at each 125 m. The simulation computes the distances from the lightning locations to the sensors and selects the events that are in the detection range, using the DE simulation mechanism for every sensor. In total there were 2984249 events in this simulation.

The trilateration used in the initial solution was simulated as a standalone algorithm, for comparison of how good/bad an iterative method is when compared to a simple algorithm.

For convergence criterion it was chosen  $\varepsilon = 0.01$  for all algorithms, i.e., gain in (3) less than 0.01 will indicate that convergence is achieved. This is somewhat arbitrary but it worked well for GNA and LMA. Other criteria were tested for SGD but with no success. For iteration criterion it was chosen  $k_{max} = 100$  for all algorithms.

The simulation computes the estimated event position, the location errors from the real positions, iterations to reach convergence and GDOP of the solutions, as shown in the results section.

For convergence it was adopted a practical basis, as described below:

- Good convergence is when the algorithm reaches convergence and the final position is less than 5 km from the real position.
- Bad convergence is when the algorithm reaches convergence but the final estimated position is greater than 5km from the real position.
- Divergence is when the final estimated position is far away from the initial solution, relative to the real position, i.e, the algorithm took the solution away from the real position.

# IV. RESULTS

# A. Network Detection Efficiency (NDE) map

DE simulation applied to the lightning area gives the NDE map shown in Fig. 2.



Fig. 2. Network Detection Efficiency (NDE)

The dark area is where the NDE reaches its maximum value, which is 100%. The area surrounded by a white line is the area where the NDE is at least 95% of the maximum value, covering 4347.78 km<sup>2</sup>, which corresponds to 45.50% of the lightning area (9554.56 km<sup>2</sup>).

#### B. Convergence

The convergence criteria applied to the algorithm's outputs are shown in Fig. 3.

LMA has the best performance, followed by GNA, when compared to SGD and TRL. Even events in bad convergence are not too far from 5 km in location error. Indeed, SGD has the worst performance from all simulated algorithms, putting most events in the



Fig. 3. Convergence results for the implemented algorithms

divergence class, probably because it needs more iterations to converge than defined in  $k_{max}$ . TRL, although is the initial solution provider, performs poorly (43.72% of the events in good convergence class), but GNA and LMA can achieve better solutions even with a poorly located initial solution. GNA performed 87.63% and LMA 112.04% better than TRL, indicating that both can be good candidates for the lightning location algorithm.

The iterative methods differs in the number of iterations it needs to converge, given as:

- SGD: 100 iterations in average.
- GNA: 48 iterations in average.
- LMA: 15 iterations in average.

Using these two aspects, the best candidate is LMA, although GNA is quite the same in good convergence, it is about 3 times slower than LMA, making the last the selected algorithm for lightning location algorithm.

#### C. Location Accuracy (LA)

A location error histogram in km of the outputs from the LMA algorithm is shown in Fig. 4. LMA has a 90% confidence level of 1.925 km, a mean error of 1.0607 km with a standard deviation of 0.7232 km, a great improvement compared to these parameters using one SD device - a mean error of -0.26 km, with a standard deviation of 1.84 km and a 90% confidence level of 3.0 km [1] - i.e, an improvement of  $\approx$  36% in LA.

In Fig. 5, GDOP works as a ceiling limit for location accuracy, i.e, a value of  $\xi(x^*) \le$  1.8 limits the location accuracy to 3km.

The RLO values are shown in TABLE I. The RLO radius  $\phi(c)$  is 1.925 km at 90% confidence level, but due to the skewness of the location error distribution, it is



Fig. 4. Location error histogram of LMA



Fig. 5. GDOP x location error of LMA

Confidence Level (c in %)	$\phi(c)$ (km)	W(c) (km <sup>2</sup> )
50	0.906	2.578
55	0.980	3.016
60	1.059	3.523
65	1.145	4.122
70	1.242	4.847
75	1.354	5.762
80	1.490	6.970
85	1.666	8.723
90	1.925	11.641
95	2.425	18.476
100	5	78.540

TABLE I RLO VALUES BY CONFIDENCE LEVEL OF LMA

also possible to select lower values of confidence level, with a great radius reduction, depending on the action taken based on the RLO. For example, a reduction of 20% in the confidence level means a reduction of  $\approx 35\%$  in  $\phi(c)$  and as a consequence the reduction of  $\approx 58\%$  in the search area W(c).

# V. DISCUSSION AND CONCLUSIONS

The choice of an iterative algorithm to handle multilateration tasks described as an optimization problem is, in fact, for a simple reason: the possibility to increase/decrease the measurements from the solution using the same workflow, i.e., without any complicated conditions to evaluate. The algorithms are simple, easy to understand and reasonably fast.

There are 3 algorithms tested in this work: SGD, GNA and LMA. Except for SGD, the performances are comparable, but in execution time, LMA works 3 times faster than GNA, although it is the heaviest in terms of operations per iteration. The tradeoff between SGD and GNA that LMA provides works faster and better than the best of the others. There are 13.01 % more events for LMA in good convergence class than for GNA.

Although the SD device has a reasonable LA for domestic applications, the simulated network based on SD parameters [1] indicates that an SD network improves LA in  $\approx 36\%$ , which is a great achievement considering a sensor designed for early warnings, which manufacturer [3] states that there is no guarantee about the lightning location. LMA has a skewed location error distribution, and RLO analysis shows a non-linear confidence level/RLO relation, indicating that a 90% confidence level is maybe too cautious for RLO purposes.

As a quality factor, GDOP is a good parameter and its values show that the proposed network geometry is on the right path. The minimum value of GDOP is 1, but it can be lower than 1 if there are many stations in solution [25]. NDE is almost 100% in 45.50 % of the lightning area. Considering that the GDOP limit is 1.8 and the NDE value, the network geometry worked well and is a good start for our short-range lightning location network.

Although it worked well, LMA can be optimized by tweaking the  $\nu$ ,  $\beta$ ,  $\gamma$  and  $\rho$  values to compute  $\mu_k$ . This is a challenging task and is a matter of further studies. The use of

Python DEAP [29] is considered for such task in the future. As for now, LMA has the best performance and is the chosen option to be the multilateration algorithm.

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